TD

Mécanique I

S1

TD intégrale 2012/2013

TD 1 : Ex 1 :

① $\vec{A} = 3\vec{j} + \vec{k}$ $\qquad \vec{B} = -2\vec{i} + 3\vec{j} + \vec{k}$

$\vec{A}.\vec{B} = x_1 x_2 + y_1 y_2 + z_1 z_2$

$= 0 + 9 + 1$

$= 10$

$\vec{A}.\vec{B} = \|\vec{A}\| \times \|\vec{B}\| \times \cos(\widehat{\vec{A},\vec{B}})$

$\cos(\widehat{\vec{A},\vec{B}}) = \dfrac{\vec{A}.\vec{B}}{\|\vec{A}\|.\|\vec{B}\|} \Rightarrow (\widehat{\vec{A},\vec{B}}) = 32,31.$

* si $\vec{A} \neq \vec{0}$ $\quad \vec{B} \neq \vec{0}$

$\vec{A}.\vec{B} = 0 \Rightarrow \vec{A} \perp \vec{B}$

Rq : $r(\theta) = \dfrac{a}{\sin\theta}$, a = cte

$\dot{r} = \dfrac{-a\cos\theta}{\sin^2\theta}$.

$\dot{r} = \left(\dfrac{1}{f}\right)' = \dfrac{-f'}{f^2}$

P.S : projection de vecteurs.

$\vec{OM} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{OM}.\vec{j} = y$

$\vec{OM}.\vec{i} = x$

$\vec{OM}.\vec{k} = z$

$\vec{OM} = (\vec{OM}.\vec{i})\,\vec{i} + (\vec{OM}.\vec{j})\,\vec{j} + (\vec{OM}.\vec{k})\,\vec{k}$

② $\vec{U}_A \begin{pmatrix} \cos\alpha_A = \dfrac{x_A}{\|\vec{A}\|} = \dfrac{0}{\sqrt{10}} = 0 \\ \cos\beta_A = \dfrac{y_A}{\|\vec{A}\|} = 3/\sqrt{10} \\ \cos\gamma_A = \dfrac{z_A}{\|\vec{A}\|} = \dfrac{1}{\sqrt{10}} \end{pmatrix}$ $\qquad \vec{U}_B \begin{pmatrix} \cos\alpha_B = \dfrac{x_B}{\|\vec{B}\|} = \dfrac{-2}{\sqrt{14}} \\ \cos\beta_B = \dfrac{y_B}{\|\vec{B}\|} = \dfrac{3}{\sqrt{14}} \\ \cos\gamma_B = \dfrac{z_B}{\|\vec{B}\|} = \dfrac{1}{\sqrt{14}} \end{pmatrix}$

$$\vec{i} \wedge \vec{j} = \vec{k}$$
$$\vec{j} \wedge \vec{k} = \vec{i}$$
$$\vec{k} \wedge \vec{i} = \vec{j}$$

3) $\vec{C} = \vec{A} \wedge \vec{B} = \begin{vmatrix} 0 & -2 \\ 3 & 3 \\ 1 & 1 \end{vmatrix} = 0\vec{i} - 2\vec{j} + 6\vec{k}$

4) $\|\vec{C}\| = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10} = 6,32$

$\|\vec{C}\| = \|\vec{A}\| . \|\vec{B}\| \sin(\vec{A}, \vec{B}) = \sqrt{40} = 6,36$.

* PV. anti commutatif.

$\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$

$\vec{A} \wedge (\vec{B} + \vec{C}) = \vec{A} \wedge \vec{B} + \vec{A} \wedge \vec{C}$

si $\vec{A} \neq \vec{0}$, $\vec{B} \neq \vec{0}$

$\vec{A} \wedge \vec{B} = \vec{0} \Rightarrow \vec{A} \parallel \vec{B}$.

si $\vec{C} = \vec{A} \wedge \vec{B}$

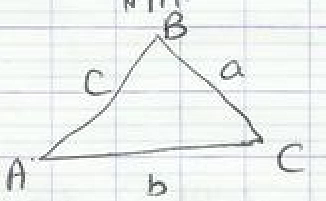
alors : $\vec{C} \perp \vec{A}$ et $\vec{C} \perp \vec{B}$

$\vec{C} \perp$ au plan formé par $\vec{A}$ et $\vec{B}$.

$\|\vec{A} \wedge \vec{B}\| = S =$ surface parallélogramme formé par $\vec{A}$ et $\vec{B}$

$S = h \|B\| = \underbrace{\|\vec{A}\| . \|\vec{B}\|}_{\sin\theta}$

$\sin\theta = \dfrac{h}{\|\vec{A}\|}$.



$a = \|\vec{BC}\|$

$b = \|\vec{AC}\|$

$c = \|\vec{AB}\|$.

$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}}$$

$(\vec{A}, \vec{B}, \vec{C}) = (\vec{C}, \vec{A}, \vec{B}) = \vec{C} . (\vec{A} \wedge \vec{B}) = \vec{C} . \vec{C} = \|\vec{C}\|^2 = 40$

$= \vec{A} . (\vec{B} \wedge \vec{C}) = 40$.

$(\vec{A}, \vec{B}, \vec{C}) = -(\vec{A}, \vec{C}, \vec{B})$

$(\vec{A}, \vec{A}, \vec{B}) = 0$ $\quad (\vec{A}, \vec{B}, \vec{C}) =$ volume du parallélépipède

TD, Ex1 (6) $\vec{A}\wedge(\vec{B}\wedge\vec{C}) = (\vec{A}\cdot\vec{C})\cdot\vec{B} - (\vec{A}\cdot\vec{B})\cdot\vec{C} = -10\,\vec{C}$

or $\vec{A} \perp \vec{C}$ donc $\vec{A}\cdot\vec{C} = 0$

$$-10\,\vec{C} = -10\begin{pmatrix} 0 \\ -2 \\ 6 \end{pmatrix} = 20\,\vec{j} - 60\,\vec{k}.$$

Ex2:

1) $\overrightarrow{A_1A_2} = (2\sqrt{2}, 4, 5)$

$\overrightarrow{A_3A_4} = \left(0, \frac{-5}{\sqrt{41}}, \frac{\lambda}{\sqrt{41}}\right)$

2) a. $\overrightarrow{A_1A_2}\cdot\overrightarrow{A_3A_4} = 0$

$$0 - \frac{5}{\sqrt{41}}\cdot 4 + \frac{5\lambda}{\sqrt{41}} = 0$$

$$\lambda = 4$$

b. $\|\overrightarrow{A_1A_2}\|^2 = 8 + 16 + 25 = 49$

$\|\overrightarrow{A_1A_2}\| = 7.$

$\|\overrightarrow{A_3A_4}\|^2 = \dfrac{25+16}{41} = 1$

c. $\vec{C} = \overrightarrow{A_1A_2}\wedge\overrightarrow{A_3A_4} = \begin{pmatrix} 2\sqrt{2} \\ 4 \\ 5 \end{pmatrix}_R \wedge \begin{pmatrix} 0 \\ -5/\sqrt{41} \\ 4/\sqrt{41} \end{pmatrix}_R$

$$= \frac{41}{\sqrt{41}}\,\vec{i} - \frac{8\sqrt{2}}{\sqrt{41}}\,\vec{j} - \frac{10\sqrt{2}}{\sqrt{41}}\,\vec{k}.$$

d. $\|\vec{C}\|^2 = \dfrac{41 + 128 + 200}{41} = 49.$

$\|\vec{C}\| = 7$

$\|\vec{C}\| = \|\overrightarrow{A_1A_2}\| \times \|\overrightarrow{A_3A_4}\| \sin\left(\frac{\pi}{2}\right) = 7 \times 1 \times 1 = 7$

e. $\vec{u}_c = \dfrac{\vec{C}}{\|\vec{C}\|} = \dfrac{\sqrt{41}}{7}\,\vec{i} - \dfrac{8\sqrt{2}}{7\sqrt{41}}\,\vec{j} - \dfrac{10\sqrt{2}}{7\sqrt{41}}\,\vec{k}.$

Ex3: $\vec{A} = 3t^2\,\vec{j} + t\,\vec{k}$ $\quad\vec{B} = -2\,\vec{i} + 3t^3\,\vec{j}$

$\vec{A}\cdot\vec{B} = 9t^5$

$$\cos\varphi = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\cdot\|\vec{B}\|} = \frac{9t^5}{\sqrt{9t^4 + t^2}\,\sqrt{4 + 9t^6}}$$

2) $$\vec{u}_A = \begin{pmatrix} \cos\alpha_A = 0 \\ \cos\beta_A = \dfrac{3t^2}{\sqrt{9t^4+t^2}} \\ \cos\gamma_A = \dfrac{t}{\sqrt{9t^4+t^2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{3t}{\sqrt{9t^2+1}} \\ \dfrac{1}{\sqrt{9t^2+1}} \end{pmatrix}_R$$

$$\vec{u}_B = \begin{pmatrix} \dfrac{-2}{\sqrt{4+9t^6}} \\ \dfrac{3t^3}{\sqrt{4+9t^6}} \\ 0 \end{pmatrix}$$

3) $\vec{C} = \vec{A} \wedge \vec{B}$

$$\begin{pmatrix} 0 \\ 3t^2 \\ t \end{pmatrix}_R \wedge \begin{pmatrix} -2 \\ 3t^3 \\ 0 \end{pmatrix}_R = \begin{pmatrix} -3t^4 \\ -2t \\ 6t^2 \end{pmatrix}_R$$

4) $\|\vec{C}\|^2 = 9t^8 + 4t^2 + 36t^4$

$\|\vec{C}\| = t\sqrt{9t^6 + 1 + 36t^2}$

5) $(\vec{A}, \vec{B}, \vec{C}) = (\vec{C}, \vec{A}, \vec{B}) = \vec{C} \cdot \vec{C} = \|\vec{C}\|^2$

6) 1er methode.

$\vec{A} \wedge (\vec{B} \wedge \vec{C})$

$$\vec{B} \wedge \vec{C} = \vec{D} = \begin{pmatrix} -2 \\ 3t^3 \\ 0 \end{pmatrix}_R \wedge \begin{pmatrix} -3t^4 \\ -2t \\ 6t^2 \end{pmatrix}_R$$

$$\vec{D} = \begin{pmatrix} 18t^5 \\ 12t^2 \\ 4t + 9t^7 \end{pmatrix}_R, \quad \vec{A} \wedge \vec{D} = \begin{pmatrix} 0 \\ 3t^2 \\ t \end{pmatrix}_R \wedge \begin{pmatrix} 18t^5 \\ 12t^2 \\ 4t + 9t^7 \end{pmatrix}_R$$

$$\vec{A} \wedge \vec{D} = \begin{pmatrix} 27t^9 \\ 18t^6 \\ -54t^7 \end{pmatrix}$$

$$\vec{A} \wedge (\vec{B} \wedge \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B}) \cdot \vec{C}$$

$$= -(\vec{A} \cdot \vec{B}) \cdot \vec{C} = -9t^5 \begin{pmatrix} -3t^4 \\ -2t \\ 6t^2 \end{pmatrix}_R$$

$$= -9t^5 \begin{pmatrix} -3t^4 \\ -2t \\ t^2 \end{pmatrix}_R .$$

Ex 4 :

$$\vec{V_1} = \begin{pmatrix} 2t \\ t \\ 1 \end{pmatrix}_R \qquad \vec{V_2} = \begin{pmatrix} 4t \\ -t \\ -t \end{pmatrix}_R$$

$$\vec{V_1} \cdot \vec{V_2} = -8t^2 - t^2 - t = 7t^2 - t$$

$$\frac{d}{dt}\left(\vec{V_1} \cdot \vec{V_2}\right) = 14t + 1$$

$$\left(\frac{d}{dt}\vec{V_1}\right) \cdot \vec{V_2} + \vec{V_1} \cdot \left(\frac{d\vec{V_2}}{dt}\right)_R$$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4t \\ -t \\ -t \end{pmatrix}_R + \begin{pmatrix} 2t \\ t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}_R$$

$$8t - t + 8t - t - 1 = 14t + 1$$

$$\vec{V_1} \wedge \vec{V_2} = \begin{pmatrix} 2t \\ t \\ 1 \end{pmatrix}_R \wedge \begin{pmatrix} 4t \\ -t \\ -t \end{pmatrix}_R$$

$$= \begin{pmatrix} -t^2 + t \\ 4t + 2t^2 \\ -6t^2 \end{pmatrix}_R$$

$$\frac{d}{dt}\left(\vec{V_1} \wedge \vec{V_2}\right)_R = \begin{pmatrix} -2t + 1 \\ 4 + 4t \\ -12t \end{pmatrix}_R = \left(\frac{d\vec{V_1}}{dt}\right) \wedge \vec{V_2} + \vec{V_1} \wedge \left(\frac{d\vec{V_2}}{dt}\right)$$

$$= \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}_R \wedge \begin{pmatrix} 4t \\ -t \\ -t \end{pmatrix}_R + \begin{pmatrix} 2t \\ t \\ 1 \end{pmatrix}\right] = \begin{pmatrix} -t - 1 + 1 \\ 2t + 4t + 2t \\ -6t - 6t \end{pmatrix} = \begin{pmatrix} -2t + 1 \\ 4t + 4 \\ -12t \end{pmatrix}$$

c- $$\frac{d}{dt}\left(\lambda \cdot \vec{V_1}\right) = \dot{\lambda}\vec{V_1} + \lambda \frac{d\vec{V_1}}{dt}$$

$$= 2e^{2t}\begin{pmatrix} 2t \\ t \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = e^{2t}\begin{pmatrix} 4t + 2 \\ 2t + 1 \\ 2 \end{pmatrix}_R$$

$$\lambda \vec{V_1} = \begin{pmatrix} 2t e^{2t} \\ t e^{2t} \\ e^{2t} \end{pmatrix} , \quad \frac{d}{dt} \lambda \vec{V_1} = e^{2t} \begin{pmatrix} 2+4t \\ 1+2t \\ 2 \end{pmatrix}$$

Ex 5 :

$$\vec{e_r} = \cos\theta \, \vec{i} + \sin\theta \, \vec{j}$$

$$\left(\frac{d\vec{e_r}}{dt}\right)_{R_1} = \left(\frac{d}{d\theta}\vec{e_r}\right)_{R_1} \cdot \frac{d\theta}{dt} = \dot{\theta}\left[-\sin\theta \, \vec{i} + \cos\theta \, \vec{j}\right] = \dot{\theta}\vec{e_\theta}$$

$$\Delta \; \frac{d}{dt}\left(\vec{X}\right)_{R_1} = \frac{d}{dt}\left(\vec{X}\right)_{R^*} + \vec{\Omega}\left(R^*/R_1\right) \wedge \vec{X}$$

$$\forall R^a \quad \left(\frac{d}{dt}\vec{e_r}\right)_{R_1} = \underbrace{\left(\frac{d\vec{e_r}}{dt}\right)_{R_2}}_{0} + \vec{\Omega}\left(R_2/R_1\right) \wedge \vec{e_r}$$

$$= \vec{0} + \vec{\Omega}\left(R_2/R_1\right) \wedge \vec{e_r}$$

$$= \dot{\theta}\vec{k} \wedge \vec{e_r} = \dot{\theta}\vec{e_\theta}$$

$$\left(\frac{d\vec{e_r}}{dt}\right)_{R_1} = \dot{\theta}\vec{e_\theta} = \vec{\Omega}\left(R_2/R_1\right) \wedge \vec{e_r}$$

$$\dot{\theta}\vec{k} \wedge \vec{e_r} = \vec{\Omega}\left(R_2/R_1\right) \wedge \vec{e_r}$$

$$\vec{\Omega}\left(R_2/R_1\right) = \dot{\theta}\vec{k}$$

$$\frac{d}{dt}\left(\vec{OM}\right)_{R_1} = \frac{d}{dt}\left(\vec{OM}\right)_{R_2} + \vec{\Omega}\left(R_2/R_1\right) \wedge \vec{OM}$$

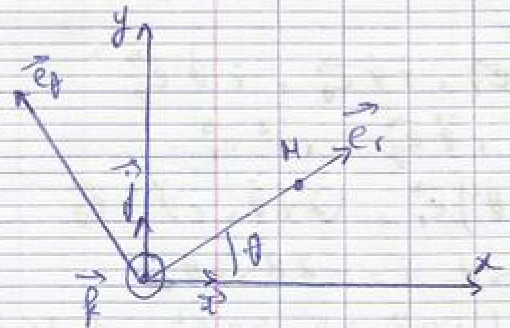
$$\vec{OM} = r\vec{e_r} + z\vec{e_z}$$

$$\frac{d}{dt}\left(\vec{OM}\right)_{R_1} = \dot{r}\vec{e_r} + \dot{z}\vec{e_z} + \dot{\theta}\vec{k} \wedge \left(r\vec{e_r} + z\vec{e_z}\right)$$

$$= \dot{r}\vec{e_r} + \dot{z}\vec{e_z} + r\dot{\theta}\vec{e_\theta}$$

SMIA 2

Ch II :



$M(r, \theta, z)$ cylindrique

si $z = 0$ coord polaires

$\vec{e_r} = \frac{\vec{OM}}{\|\vec{OM}\|}$ $r = \|\vec{OM}\|$

$\vec{e_\theta} \perp \vec{e_r}$

$\vec{e_\theta} = \vec{e_r}(\theta + \frac{\pi}{2})$

$\vec{f}(t)$

$\vec{e_r} = (\vec{e_r}\cdot\vec{i})\vec{i} + (\vec{e_r}\cdot\vec{j})\vec{j}$

$R(\vec i, \vec j, \vec k) \to R_C(\vec{e_r}, \vec{e_\theta}, \vec{e_z} = \vec{k})$

$\vec{e_r} = \cos\theta\, \vec i + \sin\theta\, \vec j$

$\|\vec{e_r}\| = 1$

• $\vec{e_\theta} = \vec{e_r}(\theta + \frac{\pi}{2}) = \cos(\theta + \frac{\pi}{2})\vec i + \sin(\theta + \frac{\pi}{2})\vec j$

$= -\sin\theta\, \vec i + \cos\theta\, \vec j$

• $(\vec{e_r}, \vec{e_\theta}, \vec k)$: $\vec k \wedge \vec{e_r} = \vec{e_\theta}$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}_R = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}_R$$

Ex 3 :

$\vec{V_0} = V_0 \vec i$

$\vec a(M) = k(\vec V(M))\vec i$

• $(\vec V)^2 = \vec V \cdot \vec V = V^2$

$\vec{OM} = x\vec i$, $\vec V(M) = \dot x \vec i$, $\vec a(M) = \ddot x \vec i$

$\frac{d\vec V}{dt} = \frac{dV}{dt}\vec i = -kV^2 \vec i$

$\frac{dV}{dt} = -kV^2 \Leftrightarrow \frac{dV}{V^2} = -k\,dt$

$f(V)\,dV = g(t)\,dt$

$\frac{-1}{V} = -kt + cte$

à $t = 0$ $V = V_0$

$-\frac{1}{V_0} = 0 + cte \Rightarrow cte = \frac{-1}{V_0}$

$\frac{1}{V} = kt + \frac{1}{V_0} \Rightarrow \frac{1}{V} = \frac{ktV_0 + 1}{V_0}$

$V(t) = \frac{V_0}{1 + ktV_0} = \dot x(t) = \frac{dx}{dt}$

$x(t)$? $\frac{dx}{dt} = \frac{V_0}{1 + kV_0 t} \Rightarrow dx = \frac{V_0\,dt}{1 + kV_0 t}$

$x(t) = \frac{1}{k}\int \frac{kV_0\,dt}{1 + kV_0 t} = \frac{1}{k}\ln(1 + kV_0 t) + cte$

$x(t) = \frac{1}{k}\ln(1 + kV_0 t)$

3) $V(x) = V_0 e^{-kx}$

$V(t) = \frac{V_0}{1 + ktV_0} \Rightarrow \frac{V_0}{V} = 1 + ktV_0$

$x = \frac{1}{k}\ln(V_0/V)$

$kx = \ln\frac{V_0}{V} \Rightarrow e^{kx} = \frac{V_0}{V}$

$r = \frac{r_0}{2}(1+\cos\theta)$ ← cardioïde

$\frac{V}{V_0} = e^{-kx}$

$\Rightarrow V = V_0\, e^{-kx}$

Exh:

$r = \|\vec{OM}\|$ $\quad \vec{e_r} = \frac{\vec{OM}}{\|\vec{OM}\|}$

$\theta = (\vec{Ox}, \vec{OM})$ $\quad \vec{e_\theta} \perp \vec{e_r}$

$\cos\theta = \frac{r}{2R} \Rightarrow r = 2R\cos\theta$

$x(\theta)$? et $y(\theta)$ $\quad f(x,y) = 0$

$\vec{OM} = r\vec{e_r} = x\vec{i} + y\vec{j}$

$= 2R\cos\theta\,[\cos\theta\,\vec{i} + \sin\theta\,\vec{j}]$

$x = 2R\cos^2\theta = R[1+\cos 2\theta]$

$y = 2R\sin\theta\cos\theta = R\sin 2\theta$

$x - R = R\cos 2\theta$

$y = R\sin 2\theta$

$(x-R)^2 + y^2 = R^2$

le cercle de centre $(R,0)$ et Rayon R.

2) $\vec{OM} = r\vec{e_r}$

$\vec{V}(M)_R = \frac{d}{dt}(\vec{OM})_R = \frac{d}{dt}[r\vec{e_r}]_R$

$= \dot{r}\vec{e_r} + r\dot\theta\,\vec{e_\theta}$

$r = 2R\cos\theta$, $\dot r = 2R\dot\theta\sin\theta$

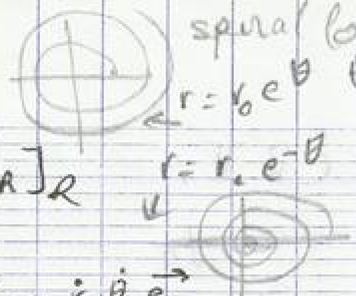
$\vec{V} = (-2R\dot\theta\sin\theta\,\vec{e_r} + 2R\dot\theta\cos\theta\,\vec{e_\theta})$

$= V_r\vec{e_r} + V_\theta\vec{e_\theta}$

$V_r = -2R\dot\theta\sin\theta$

$V_\theta = 2R\dot\theta\cos\theta$.

conique



$\vec\Gamma(M)_R = \frac{d}{dt}[\vec{V}(M)_R]_R$

$= \ddot r\vec{e_r} + \dot r\dot\theta\,\vec{e_\theta} + \dot r\dot\theta\,\vec{e_\theta} + r\ddot\theta\,\vec{e_\theta} - r\dot\theta^2\vec{e_r}$.

$= (\ddot r - r\dot\theta^2)\vec{e_r} + (2\dot r\dot\theta + r\ddot\theta)\vec{e_\theta}$

$r = 2R\cos\theta \quad \dot r = -2R\sin\theta$

$\vec\Gamma(M)_R = (-2R\ddot\theta\sin\theta - 2R\dot\theta^2\cos\theta - 2R\dot\theta^2\cos\theta)\vec{e_r}$

$+ (-4R\dot\theta^2\sin\theta + 2R\ddot\theta\cos\theta)\vec{e_\theta}$.

$= (-2R\ddot\theta\sin\theta - 4R\dot\theta^2\cos\theta)\vec{e_r}$

$+ (-4R\dot\theta^2\sin\theta + 2R\ddot\theta\cos\theta)\vec{e_\theta}$

3) a- $\frac{ds}{dt} = v \quad ds = \sqrt{(dr)^2 + (r\,d\theta)^2}$

$ds = v\,dt$

$\vec V = 2R\dot\theta[-\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}]$

$v = \|\vec V\| = 2R\dot\theta$

$ds = 2R\dot\theta\,dt = 2R\,d\theta$

$s = 2R\theta + cte$

à $t=0$ $\quad \theta = 0 \Rightarrow cte = 0$

$\boxed{s = 2R\theta}$

b- $(\vec\tau, \vec n, \vec b = \vec\tau \wedge \vec n)$

$\vec\tau = \frac{\vec V}{\|\vec V\|} = \frac{2R\dot\theta[-\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}]}{2R\dot\theta}$

$\vec\tau = -\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}$.

$\vec V = 2R\dot\theta\,\vec\tau$.

$\vec\Gamma(M)_R = (-2R\ddot\theta\sin\theta - 4R\dot\theta^2\cos\theta)\vec{e_r}$

$+ (-4R\dot\theta^2\sin\theta + 2R\ddot\theta\cos\theta)\vec{e_\theta}$

1er méthode:

$\vec\Gamma(M)_R = \frac{d}{dt}(2R\dot\theta\,\vec\tau)$

$= 2R\ddot\theta\,\vec\tau + 2R\dot\theta\left(\frac{d\vec\tau}{dt}\right)$

$\frac{d\vec\tau}{dt} = \frac{\vec n}{R_c}V$ | $\vec\Gamma(M)_R = 2R\ddot\theta\,\vec\tau + \frac{(2R\dot\theta)2R\dot\theta}{R}\vec n$

$= 2R\ddot\theta\,\vec\tau + 4R\dot\theta^2\,\vec n$

Ex 18 sphérique

$2^{ème}$ méthode :

$$\vec{\Gamma}(M)_R = 2R\ddot{\theta}[-\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}] + 4R\dot{\theta}^2[-\cos\theta\,\vec{e_r} - \sin\theta\,\vec{e_\theta}]$$

$$= 2R\ddot{\theta}\,\vec{\tau} + 4R\dot{\theta}^2\,\vec{n}$$

$$\vec{n} = \frac{\frac{d\vec{\tau}}{dt}}{\left\|\frac{d\vec{\tau}}{dt}\right\|}$$ Générale $\forall$ le mouvement

$\vec{n} = \vec{b} \wedge \vec{\tau}$ (Mouv plan)

$\vec{b} = \pm\vec{k}$

Ici $\vec{b} = \vec{k}$

$$\vec{n}\begin{pmatrix}0\\0\\1\end{pmatrix}_{(\vec{e_r},\vec{e_\theta},\vec{k})} \wedge \begin{pmatrix}-\sin\theta\\\cos\theta\\0\end{pmatrix}_{(\vec{e_r},\vec{e_\theta},\vec{k})}$$

$\vec{\tau} = -\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}$

$$\frac{d\vec{\tau}}{dt} = -\dot{\theta}\cos\theta\,\vec{e_r} - \dot{\theta}\sin\theta\,\vec{e_\theta} - \dot{\theta}\sin\theta\,\vec{e_\theta} - \dot{\theta}\cos\theta\,\vec{e_r}$$

$$= -2\dot{\theta}\cos\theta\,\vec{e_r} - 2\dot{\theta}\sin\theta\,\vec{e_\theta}$$

$$\left\|\frac{d\vec{\tau}}{dt}\right\| = 2\dot{\theta}$$

$$\frac{\frac{d\vec{\tau}}{dt}}{\left\|\frac{d\vec{\tau}}{dt}\right\|} = -\cos\theta\,\vec{e_r} - \sin\theta\,\vec{e_\theta}$$

$$\frac{d\vec{\tau}}{dt} = \frac{\vec{n}}{R_c}V \Rightarrow \left\|\frac{d\vec{\tau}}{dt}\right\| = \frac{V}{R_c}$$

$$\cancel{2\dot{\theta}} = \frac{2R\dot{\theta}}{R_c} \Rightarrow R_c = R$$

CP. $\dot{\theta} = w \Rightarrow \theta(t) = w(t) + \theta_0$

$\theta(t) = wt \qquad \theta_0 = 0$

$r = 2R\cos\theta = 2R\cos wt$

$\vec{V} = 2R\dot{\theta}\,\vec{\tau} = 2Rw\,\vec{\tau}$

$\vec{\Gamma} = 2R\ddot{\theta}\,\vec{\tau} + 4R\dot{\theta}^2\,\vec{n}$

$\dot{\theta} = w \qquad \ddot{\theta} = 0 \qquad \vec{\Gamma} = 4Rw^2\vec{n}$.

Ex 6 :

$$\begin{cases} x(t) = a\cos\varphi \\ y(t) = a\sin\varphi \\ z = h\varphi \end{cases}$$

$$\begin{cases} x^2 + y^2 = a^2 \\ z = h\varphi \end{cases}$$

$\vec{OM} = \rho\vec{e_\rho} + z\vec{e_z}$

$= x\vec{i} + y\vec{j} + z\vec{k}$

$= a\cos\varphi\,\vec{i} + a\sin\varphi\,\vec{j} + h\varphi\,\vec{k}$

$\vec{OM} = a[\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}] + h\varphi\,\vec{k}$

$\vec{OM} = \rho[\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}] + h\varphi\,\vec{k}$

$$\begin{cases} \rho = a = cte \\ \varphi(t) \\ z = h\varphi. \end{cases}$$

$\vec{OM} = a\vec{e_\rho} + h\varphi\,\vec{e_z}$

1) $\vec{V}(M/R) = \frac{d}{dt}(\vec{OM})_R = \frac{d}{dt}[\rho\vec{e_\rho} + z\vec{e_z}]_R$

$= \dot{\rho}\vec{e_\rho} + \rho\dot{\varphi}\vec{e_\varphi} + \dot{z}\vec{e_z}$

ici : $\rho = a = cte \Rightarrow \dot{\rho} = 0$

$\vec{V}(M/R) = a\dot{\varphi}\,\vec{e_\varphi} + h\dot{\varphi}\,\vec{e_z}$

$= \dot{\varphi}\,[a\vec{e_\varphi} + h\vec{e_z}]$.

$\vec{\Gamma}(M)_R = \vec{\Gamma}(M/R)$

$= \frac{d}{dt}[\vec{V}(M)_R]_R$

$= \dot{\rho}\dot{\varphi}\vec{e_\varphi} + \rho\ddot{\varphi}\vec{e_\varphi} - \rho\dot{\varphi}^2\vec{e_\rho} + \ddot{z}\vec{e_z}$

$= (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e_\rho} + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e_\varphi} + \ddot{z}\vec{e_z}$

$\rho = a \quad \dot\rho = 0 \quad \ddot\rho = 0$

$\vec\Gamma(M)_R = -a\dot\varphi^2\vec{e}_\rho + a\ddot\varphi\,\vec{e}_\varphi + h\ddot\varphi\cdot\vec{e}_z$

si $z=0$ (ρ,φ) polaires

2) $\vec{e}_\varphi \in [xOy]$

$tg\,\alpha = \frac{h\dot\varphi}{a\dot\varphi} = \frac{h}{a} = cte$

3) a- $\dot\varphi = w \quad \ddot\varphi = 0$

$\vec\Gamma(M/R) = -a\,w^2\vec{e}_\rho$

$\mathcal{M}_{\vec{e}_z}(\vec\gamma) = (\vec{OM}\wedge\vec\gamma)\cdot\vec{e}_z = 0$

$= [(a\vec{e}_\rho + h\varphi\,\vec{e}_z)\wedge(-aw^2\vec{e}_\rho)]\,\vec{e}_z$

$= a\,h\,w^2\,\vec{e}_\rho\cdot\vec{e}_z = 0$

b- $s = \int v\,dt \qquad \vec{V} = w(h\,\vec{e}_z + a\vec{e}_\rho)$

$\|\vec{V}\| = v = w\sqrt{h^2+a^2}$

$s = w\sqrt{h^2+w^2}\int_0^t du =$

$w\sqrt{h^2+w^2}\,t = wt\sqrt{h^2+a^2} = \varphi\sqrt{a^2+h^2}$

$ds = \sqrt{(d\rho)^2 + (\rho\,d\varphi)^2 + (dz)^2}$

$ds = (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\varphi)^2$

$\vec{OM} = \rho\,\vec{e}_\rho + z\,\vec{e}_z$

$d\vec{OM} = d\rho\,\vec{e}_\rho + \rho\,d\vec{e}_\rho + dz\,\vec{e}_z$

or $\frac{d\vec{e}_\rho}{d\varphi} = \vec{e}_\varphi$

$d\rho\,\vec{e}_\rho + \rho\,d\varphi\,\vec{e}_\varphi + dz\,\vec{e}_z$

$\vec{OM} = r\vec{e}_r$

$d\vec{OM} = dr\,\vec{e}_r + r\,d\vec{e}_r(\theta,\varphi)$

$ds = \sqrt{0 + a^2(d\varphi)^2 + h^2(d\varphi)^2}$

$ds = \sqrt{a^2+h^2}\,d\varphi$

$s = \varphi\sqrt{a^2+h^2} + cte$

$s(0) = 0 \Rightarrow cte = 0$

$s(\varphi) = \varphi\sqrt{a^2+h^2}$

R_c ? $\Gamma_n = \frac{v^2}{R_c} \Rightarrow R_c = \frac{v^2}{\Gamma_n}$

$\vec\Gamma = -aw\,\vec{e}_\rho$

$v = w\sqrt{h^2+a^2}$

$\Rightarrow \Gamma_t = 0$

$\vec\Gamma = \Gamma_n\vec{n} + \Gamma_t\vec{t} = \Gamma_n\vec{n}$

$\Gamma_n = aw^2,\ \vec{n} = -\vec{e}_\rho$

$R_c = \frac{w^2(h^2+a^2)}{aw^2} = \frac{h^2+a^2}{a}$

$R_c = \frac{v^3}{\|\vec{V}\wedge\vec\Gamma\|}$

$\vec{V}\wedge\vec\Gamma = w(a\vec{e}_\rho + h\vec{e}_z)\wedge[-aw\,\vec{e}_\rho]$

$= a^2w^2\vec{e}_z - h\,a\,w^3\vec{e}_\varphi$

$= aw^3[a\vec{e}_z - h\vec{e}_\varphi]$

$\|\vec{V}\wedge\vec\Gamma\| = aw^3\sqrt{a^2+h^2}$

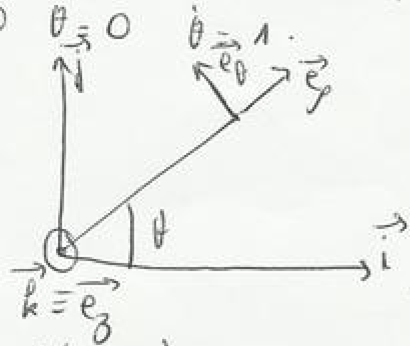
$R_c = \frac{w^3(h^2+a^2)^{3/2}}{aw^3(a^2+h^2)^{1/2}} = \frac{h^2+a^2}{a}$

Chapitre 3 :

Ex 4 :

$R_0(O, \vec{i}, \vec{j}, \vec{k})$, $R_1(O, \vec{e_\rho}, \vec{e_\theta}, \vec{e_3} = \vec{k})$

à $t = 0$ $\theta = 0$ $\dot{\theta} = 1$.



$\vec{v_a}(M) = \vec{v}(M/R_0) = \frac{d}{dt}[\vec{OM}]_{R_0}$

$\vec{OM} = \rho\vec{e_\rho} + z = 0$

$= e^\theta \vec{e_\rho}$

$\vec{v}(M/R_0) = \dot{\theta}\vec{e_\rho} + \theta e^\theta \vec{e_\theta} = \dot{\theta}e^\theta[\vec{e_\rho} + \vec{e_\theta}]$

$= (\dot{\theta}\vec{e_\rho} + \rho\dot{\theta}\vec{e_\theta})$

$\vec{v_e}(M) = \vec{v_0}(O_1) + \vec{\Omega}(R_1/R_0) \wedge \vec{O_1M}$

$\vec{v}(M \in R_1/R_0) = \vec{v}(M/R_0)$ $\left.\begin{array}{l} x_1 = cte \\ y_1 = cte \\ z_1 = cte \end{array}\right\}$ coord relative.

$\vec{V_e}(M) = \vec{v}(M/R_0)\,|\,\rho = cte = \rho\dot{\theta}\vec{e_\theta} = \dot{\theta}e^\theta\vec{e_\theta}$

$\vec{v_a}(M) = \vec{V_r}(M) + \vec{V_e}(M)$

$\vec{V_r}(M) = \vec{V_a}(M) - \vec{V_e}(M) = \dot{\rho}\vec{e_\rho} = \dot{\theta}e^\theta\vec{e_\rho}$

$\vec{V_r}(M) = \frac{d}{dt}[\vec{OM}]_{R_1}$

$\vec{V}(M/R_0) = \vec{V}(M/R_1) + \vec{V}(M \in R_1/R_0)$

$\vec{\Gamma}(M/R_0) = \frac{d}{dt}[\vec{v}(M/R_0)]_{R_0}$

$= \frac{d}{dt}[\dot{\theta}e^\theta(\vec{e_\rho} + \vec{e_\theta})]_{R_0}$

$= (\ddot{\theta}e^\theta + \dot{\theta}^2 e^\theta)(\vec{e_\rho} + \vec{e_\theta})$

$+ \dot{\theta}e^\theta(\dot{\theta}\vec{e_\theta} - \dot{\theta}\vec{e_\rho})$

$= e^\theta\{[\ddot{\theta} - \dot{\theta}^2 - \dot{\theta}^2]\vec{e_\rho} + (\ddot{\theta} + 2\dot{\theta})\vec{e_\theta}\} =$

$= e^\theta[\ddot{\theta}\vec{e_\rho} + (\ddot{\theta} + 2\dot{\theta}^2)\vec{e_\theta}]$

$\vec{\Gamma}(M/R_0) = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{e_\rho} + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{e_\theta}$

$\vec{\Gamma_e}(M) = \vec{\Gamma_a}(O_1) + \frac{d}{dt}[\vec{\Omega}(R_1/R_0)] \wedge \vec{O_1M}$

$+ \vec{\Omega}(R_1/R_0) \wedge [\vec{\Omega}(R_1/R_0) \wedge \vec{OM}]$.

$\ddot{\theta}\vec{e_3} \wedge e^\theta\vec{e_\rho} + \dot{\theta}\vec{e_3} \wedge [\dot{\theta}\vec{e_3} \wedge e^\theta\vec{e_\rho}]$

$= e^\theta[\ddot{\theta}\vec{e_\theta} - \dot{\theta}^2\vec{e_\rho}]$

$\vec{\Gamma_e}(M) = \frac{d}{dt}[\vec{V_e}(M)]_{R_0} - \underbrace{\vec{\Omega}(R_1/R_0) \wedge \vec{V_r}}_{\frac{1}{2}\Gamma_c(M) = \frac{1}{2}[2\vec{\Omega}(R_1/R_0) \wedge \vec{V_r}]}$

$\vec{\Gamma_e}(M) = -\dot{\theta}^2 e^\theta\vec{e_\rho} + \ddot{\theta}e^\theta\vec{e_\theta}$

$\vec{\Gamma_r}(M) = \frac{d}{dt}[\vec{V}(M/R_1)]_{R_1} = \frac{d}{dt}[\dot{\theta}e^\theta\vec{e_\rho}]_{R_1}$

$= (\ddot{\theta}e^\theta + \dot{\theta}^2 e^\theta)\vec{e_\rho}$

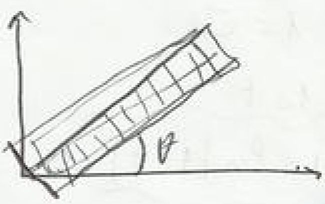
$= e^\theta(\ddot{\theta} + \dot{\theta}^2)\vec{e_\rho}$

$\vec{\Gamma_c}(M) = 2\vec{\Omega}(R_1/R_0) \wedge \vec{V_r} = 2\dot{\theta}\vec{e_3} \wedge \dot{\theta}e^\theta\vec{e_\rho}$

$= 2\dot{\theta}^2 e^\theta\vec{e_\theta}$

3) lois des aires (verifiée plan)

$\rho^2\dot{\theta} = C$



$ds = \frac{1}{2}\rho^2 d\theta$

$\frac{ds}{dt} = \frac{1}{2}\rho^2\dot{\theta}^2 = \frac{C}{2}$

Vitesse aleatoire

$\vec{OQ_1} = \vec{V}(M/R_1)$ $\vec{V}(M)_{R} \cdot \vec{V}(Q_1)_{R_0} = 0$

$\vec{OQ_1} = \vec{V}(M/R_1) = \dot{\theta}e^\theta\vec{e_\rho}$

$\vec{V}(Q/R_0) = \frac{d}{dt}(\vec{OQ})_{R_0} = e^\theta[(\ddot{\theta} + \dot{\theta}^2)\vec{e_\rho} + \dot{\theta}^2\vec{e_\theta}]$

$\begin{pmatrix} \dot{\theta}e^\theta \\ \dot{\theta}e^\theta \\ 0 \end{pmatrix}_{R_1} \cdot \begin{pmatrix} (\ddot{\theta} + \dot{\theta}^2)e^\theta \\ \dot{\theta}^2 e^\theta \\ 0 \end{pmatrix}_{R_1}$

$e^{2\theta}[\dot{\theta}\ddot{\theta} + 2\dot{\theta}^3] = 0$

$\dot{\theta}[2\dot{\theta}^2 e^{2\theta} + \ddot{\theta}e^{2\theta}] = 0$

$\dot{\theta} \neq 0$

ou $[2\dot{\theta}^2 e^{2\theta} + \ddot{\theta} e^{2\theta}] = 0$

$\frac{d}{dt}[\dot{\theta} e^{2\theta}] = 0 \Rightarrow \dot{\theta} e^{2\theta} = C$

$\|\lambda\vec{A}\| = |\lambda| \|\vec{A}\|$

$\dot{\theta}\rho^2 = C$

$\|\vec{V}_a(M)\| = cte$

or $\vec{V}_a(M) = \dot{\theta} e^{\theta}(\vec{e}_\rho + \vec{e}_\theta)$

$\vec{V}\cdot\vec{\Gamma} = 0$ uniforme

$\vec{V}\cdot\vec{\Gamma} > 0 \ ct \nearrow$ Accelération

$\vec{V}\cdot\vec{\Gamma} < 0 \ ct \searrow$ Deccelération

$\|\vec{V}(M)\| = \dot{\theta} e^{\theta}\sqrt{2} = cte$

$\|\lambda\vec{A}\| = |\lambda| \|\vec{A}\|$

$\dot{\theta} e^{\theta}\sqrt{2} = \dot{\theta}_0 e^{\theta_0}\sqrt{2} = 1 \times 1\sqrt{2} = \sqrt{2}$

$\dot{\theta} e^{\theta}\sqrt{2} = \sqrt{2}$

$\dot{\theta} e^{\theta} = 1$

$\dot{\theta} = \frac{1}{e^{\theta}} = \frac{d\theta}{dt}$

$e^{\theta} d\theta = dt$

$f(\theta)d\theta = f(t)dt$

$e^{\theta} = t + cte$

à $t=0$ $e^{\theta_0} = 0 + cte$

$1 = cte$

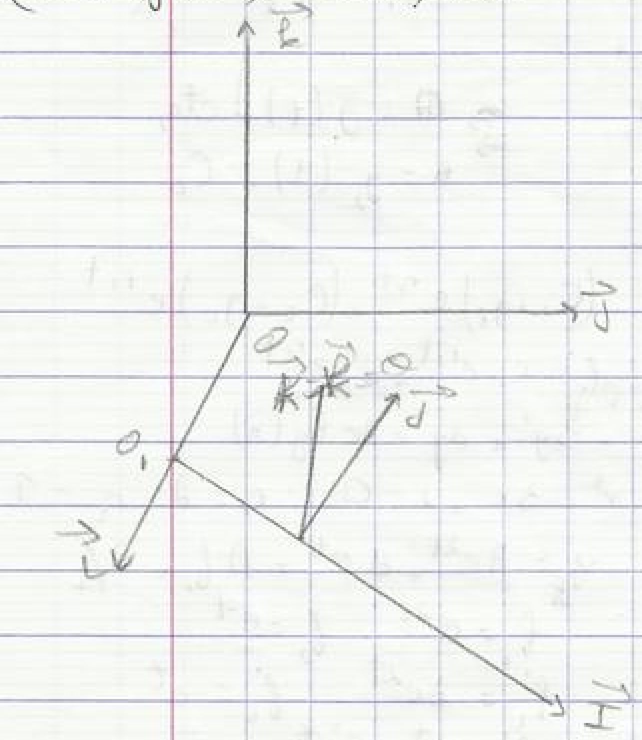
$e^{\theta} = 1 + t$

$\boxed{\theta(t) = \ln(1+t)}$

chapitre 3:

Ex 23

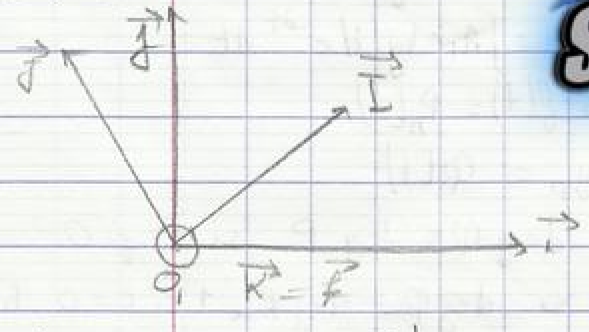
$(O,\vec i,\vec j,\vec k)$ $\quad \vec{OO_1} = \mu(t)\vec i$



$R_1(O_1,\vec I,\vec J,\vec K)$

$x_1 = a\cos 2wt$

$y_1 = -a\sin 2wt$



$\vec V_a(M) = \vec V(M/R_0) = \frac{d}{dt}(\vec{OM})_{R_0}$

$= \frac{d}{dt}(\vec{OO_1} + \vec{O_1M})_{R_0}$

$= \dot\mu\vec i + \dot x_1\vec I + x_1\dot\theta\vec J + \dot y_1\vec J - y_1\dot\theta\vec I$

$= \dot\mu\vec i + (\dot x_1 - y_1\dot\theta)\vec I + (\dot y_1 + x_1\dot\theta)\vec J$

avec $\dot x_1 = -2aw\sin 2wt$

$\dot y_1 = -2aw\cos 2wt$

$\vec V_r(M) = \vec V(M/R_1) = \frac{d}{dt}(\vec{O_1M})_{R_1}$

$= \frac{d}{dt}(x_1\vec I + y_1\vec J)_{R_1} = \dot x_1\vec I + \dot y_1\vec J$

$$\begin{cases} \vec V_e(M) = \vec V_a(M) - \vec V_r(M) \\ \vec V(M\in R_1/R_0) = \dot\mu\vec i - y_1\dot\theta\vec I + x_1\dot\theta\vec J \\ \vec V_a(O_1) + \vec\Omega(R_1/R_0)\wedge\vec{O_1M} \end{cases}$$

2) $\vec\Gamma_a(M) = \vec\Gamma(M/R_0) = \frac{d}{dt}\left[\vec V(M/R_0)\right]_{R_0}$

$= \ddot\mu\vec i + (\ddot x_1 - \dot y_1\dot\theta - y_1\ddot\theta)\vec I$

$+ (\dot x_1 - y_1\dot\theta)\dot\theta\vec J + (\ddot y_1 + \dot x_1\dot\theta + x_1\ddot\theta)\vec J$

$+ (\dot y_1 + x_1\dot\theta)(-\dot\theta\vec I)$

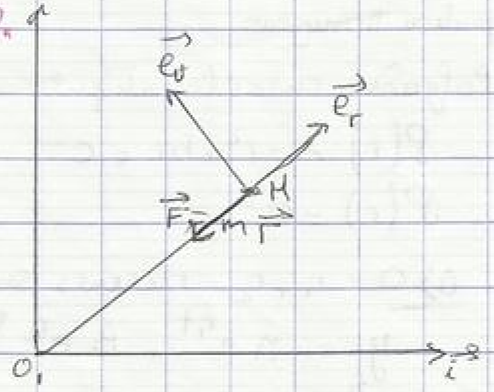
$= \ddot\mu\vec i + (\ddot x_1 - 2\dot y_1\dot\theta - y_1\ddot\theta - x_1\dot\theta^2)\vec I$

$+ (\ddot y_1 + 2\dot x_1\dot\theta + x_1\ddot\theta - y_1\dot\theta^2)\vec J$

$\ddot x_1 = -4aw^2\cos 2wt$

$\ddot y_1 = 4aw^2\sin 2wt$

Mq le mouvement relative est a acceleration centrale de centre O_1



$\vec{O_1M} \parallel \vec\Gamma_r(M)$

$\vec\Gamma_r(M) = 4aw^2\cos 2wt\,\vec I + 4aw^2\sin 2wt\,\vec J$

$\vec\Gamma_r(M) = -4w^2(a\cos 2wt\,\vec I - a\sin 2wt\,\vec J)$

$= -4w^2\vec{O_1M}$

$\vec\Gamma_e(M) = \vec\Gamma_a(O_1) + \frac{d}{dt}\left[\vec\Omega(R_1/R_0)\right]\wedge\vec{O_1M}$

$+ \vec\Omega(R_1/R_0)\wedge\left[\vec\Omega(R_1/R_0)\wedge\vec{O_1M}\right]$

$\vec\Gamma(M\in R_1/R_0) = \ddot\mu\vec i + (-y_1\ddot\theta - x_1\dot\theta^2)\vec I$

$+ (x_1\ddot\theta - y_1\dot\theta^2)\vec J$

$$= \frac{d}{dt}\left[\vec{V}_r(M)\right]_{R_0} + \underbrace{\vec{\Omega}(R_1/R_0) \wedge \vec{V}_r(M)}_{\frac{1}{2}\vec{\Gamma}_c(M)}$$

$$\vec{\Gamma}_c(M) = 2\vec{\Omega}(R_1/R_0) \wedge \vec{V}_r = 2\dot{\theta}\vec{K} \wedge (\dot{x}_1\vec{I} + \dot{y}_1\vec{J})$$
$$= 2\dot{x}_1\dot{\theta}\vec{J} - 2\dot{y}_1\dot{\theta}\vec{I}$$

3) $\vec{\Gamma}_a(M) = -w^2\vec{OM}$

$$\left(\frac{d^2\vec{OM}}{dt^2}\right) = -w^2\vec{OM}$$

si $\vec{OM} = x\vec{i} + y\vec{j}$

$$\ddot{x}\vec{i} + \ddot{y}\vec{j} = -w^2(x\vec{i} + y\vec{j})$$

$$\ddot{x} = -w^2 x \qquad \ddot{x} + w^2 x = 0$$
$$\ddot{y} = -w^2 y \iff \ddot{y} + w^2 y = 0$$

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

EDSM

Equation Homogene

Polynôme caracteristique

$P(r) = ar^2 + br + c$

$P(r) = 0$

$\Delta > 0$ r_1, r_2 racines simples distinctes

$$y_R = Ae^{r_1 t} + Be^{r_2 t}$$

$\Delta = 0$ $r_1 = r_2 = r$ racine double

$$(At + B)e^{rt}$$

$\Delta < 0$ $r_1 = \alpha + i\beta$ $r_2 = \alpha - i\beta$

$$y_R = e^{\alpha t}\left[A\cos\beta t + B\sin\beta t\right]$$
$$Ke^{\alpha t}\cos(\beta t + \varphi)$$
$$Ce^{\alpha t}\sin(\beta t + \psi)$$

Solution particulaire

$$a\ddot{y} + b\dot{y} + cy = f(t)$$

Methode variation de la constante

LAGRAGE

$$y_R = Af_1(t) + Bf_2$$

$(At + B)e^{r_1 t}$

$$\begin{cases} A'f_1 + B'f_2 = 0 \\ A'f_1' + B'f_2' = \frac{f(t)}{a} \end{cases}$$

$$\Rightarrow \begin{matrix} A = g_1(t) + cte_1 \\ B = g_2(t) + C_2 \end{matrix}$$

$$(C_1 + g_1)e^{r_1 t} + (C_2 + g_2)e^{r_2 t}$$

Exemple $= C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$$\ddot{y} - 3y' + 2y = \mathrm{Arctg}(x)$$

$r^2 - 3r + 2 = 0$ $r_1 = 2$ $r_2 = 1$

$$y_R = Ae^{2t} + Be^{t} = Af_1 + Bf_2$$

$f_1 = e^{2t}$ $f_2 = e^{-t}$

$f_1' = 2e^{2t}$ $f_2' = e^{t}$

$$\begin{cases} A'e^{2t} + B'e^{t} = 0 \\ 2A'e^{2t} + B'e^{t} = \mathrm{Arctg}(t) \end{cases}$$

$$A'e^{2t} = \mathrm{Arctg}(t)$$

$$A' = \mathrm{Arctg}(t)e^{-2t}$$

$$A = \int \mathrm{Arctg}(t)e^{-2t}\,dt$$

si $f(t) = P_n(t)$

$y_p = Q(t)$

$\deg Q = \deg P$ si $C \neq 0$

si $\deg Q = \deg P + 1$ si $C = 0$ $b \neq 0$

$d^\circ Q = d^\circ P + 2$ si $C = 0$ $b = 0$

$$-w^2\vec{OM} = \vec{\Gamma}(M/R_0)$$

$$\Rightarrow \begin{cases} \ddot{x} + w^2 x = 0 \\ \ddot{y} + w^2 y = 0 \end{cases}$$

$x(t) = A\cos wt + B\sin wt$

$y(t) = C\cos wt + D\sin wt$

or $\vec{OM} = \vec{OO_1} + \vec{O_1M}$

$$= \mu\vec{i} + x_1\vec{I} + y_1\vec{J}$$
$$= x\vec{i} + y\vec{j} + z\vec{k}$$

Chap 3 :

Ex2 :

$\vec{OM} = \mu\vec{i} + a\cos 2wt\,(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) -$

$- a\sin 2wt\,(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})$

$x = \mu + a\cos(2wt - \theta)$

$y = -a\sin(2wt - \theta)$

à $t=0$ $\quad x(0) = A \qquad A = \mu(0) + a$

$A = a$

$\dot{x}(0) = Bw = \dot{\mu}(0) = aw$

$B = a.$

$y(0) = C = 0.$

$\dot{y}(0) = Dw = -a(2w - \dot{\theta}(0)) = Dw$

$-a(2w - w) = Dw$

$-aw = Dw$

$D = -a.$

$x = a\cos wt + a\sin wt$

$= \mu + a\cos(2wt - \theta)$

$y = -a\sin wt = -a\sin(2wt - \theta)$

$2wt - \theta = wt$

$\Rightarrow \boxed{\theta(t) = wt}$

$a\cos wt + a\sin wt = \mu + a\cos wt$

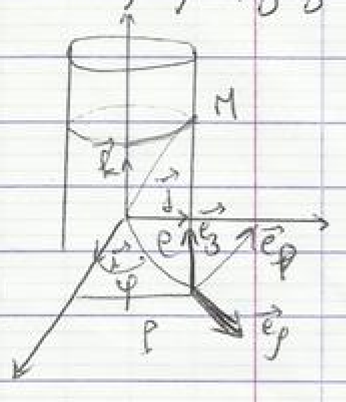
$\mu = a\sin wt$

Chapitre 4 :

Ex 3 :

$\rho = a = cte$

$\vec{OM} = \rho\,\vec{e_\rho} + z\,\vec{e_z}$



$\vec{V}(M/R) = a\dot{\varphi}\,\vec{e_\varphi} + \dot{z}\,\vec{e_z}$

$\vec{\Gamma}(M/R) = a\ddot{\varphi}\,\vec{e_\varphi} - a\dot{\varphi}^2\,\vec{e_\rho} + \ddot{z}\,\vec{e_z}$

$z(0) = z_0$ et $\dot{z}(0) = \dot{z}_0$

$\sum \vec{F}_{ext} = m\,\vec{\Gamma}(M/R)$

$\vec{P} + \vec{F} + \vec{R}$

$-mg\,\vec{e_z} + mk^2[-a\,\vec{e_\rho} - a\lambda^2\,\vec{e_\varphi} + (h - z)\,\vec{e_z}]$

$+ N\,\vec{e_\rho}$

$$\begin{cases} -ma\dot{\varphi}^2 = -mak^2 + N & (1) \\ ma\ddot{\varphi} = -ma(k\lambda)^2 & (2) \\ m\ddot{z} = mk^2(h - z) - mg & (3) \end{cases}$$

(2) $\ddot{\varphi} = -k^2\lambda^2$

$\ddot{\varphi} = -(k\lambda)^2 \Rightarrow \dot{\varphi} = -(k\lambda)^2 t + cte$

à $t=0$ $\quad \dot{\varphi} = 0 = cte = 0$

$\dot{\varphi} = -(k\lambda)^2 t \Rightarrow \varphi = -(k\lambda)^2\dfrac{t^2}{2} + cte$

$\varphi(0) = \varphi_0 \Rightarrow cte = \varphi_0$

$\boxed{\varphi(t) = -\frac{1}{2}(k\lambda)^2 t^2 + \varphi_0}$

$\ddot{z} = k^2(h - z) - g$

$a\ddot{z} + b\dot{z} + cz = f(t)$

$\ddot{z} + k^2 z = k^2 h - g$

solution de l'equation homogene

$\ddot{z} + k^2 z = 0$

$z_h = A\cos kt + B\sin kt$

solution particulière

$$\text{si } f(t) = P_n(t) \qquad z_p = Q(t) \text{ telque } \begin{cases} d^\circ Q = d^\circ P \text{ si } c \neq 0 \\ d^\circ Q = d^\circ P + 1 \text{ si } c = 0,\ b \neq 0 \\ d^\circ Q = d^\circ P + 2 \text{ si } c = 0,\ b = 0 \end{cases}$$

$z_p = cte$

$\ddot{z}_p + k^2 z_p = k^2 h - g$

$0 + k^2 z_p = k^2 h - g$

$z_p = \dfrac{k^2 h - g}{k^2} = h - \dfrac{g}{k^2}$

SMIA 2

loi des aires $\rho^2 \dot{\theta} = cte$

$$z = z_h + z_p$$
$$z = A\cos kt + B\sin kt + h - \frac{g}{k^2}$$
$$z(0) = z_0 = A + h - \frac{g}{k^2}$$
$$\Rightarrow \dot{A} = z_0 - h + \frac{g}{k^2}$$
$$\dot{z}(0) = \dot{z}_0 = Bk \Rightarrow B = \dot{z}_0/k$$
$$z(t) = \left(z_0 - h + \frac{g}{k^2}\right)\cos kt + \frac{\dot{z}_0}{k}\sin kt + h - \frac{g}{k^2}$$

$$-ma\dot{\varphi}^2 = N - mak^2$$
$$N(t) = -ma\dot{\varphi}^2 + mak^2$$
$$\varphi(t) = -(k\lambda)^2\frac{t^2}{2} + \varphi_0$$

2^{eme} partie :

$$\vec{\Gamma}(M/R) = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e}_\varphi + \ddot{z}\vec{e}_3$$

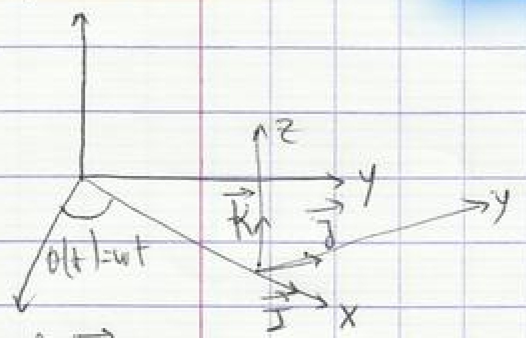
$$\sum \vec{F}_{ext} = m\vec{\Gamma}(M/R)$$
$$\begin{cases} m(\ddot{\rho} - \rho\dot{\varphi}^2) = -mk^2\rho \\ m(2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}) = 0 \\ m\ddot{z} = mk^2(h - z) - mg \end{cases}$$
$$\rho^2\dot{\varphi} = C$$
$$2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} = \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\varphi}) = 0$$

$\rho^2\dot{\varphi} = C$ loi des aires

$$\ddot{\rho} - \rho\dot{\varphi}^2 = -k^2\rho$$
$$\ddot{\rho} - \frac{C^2}{\rho^3} = -k^2\rho$$
$$\dot{\rho}\ddot{\rho} - \frac{C^2\dot{\rho}}{\rho^3} = -k^2\rho\dot{\rho}$$
$$\frac{1}{2}\dot{\rho}^2 + \frac{C^2}{2\rho^2} = -k^2\frac{\rho^2}{2} + cte$$
$$\dot{\rho}^2 + \frac{C^2}{\rho^2} = -k^2\rho^2 + K$$

$$\dot{\rho}^2 = -k^2\rho^2 - \frac{C^2}{\rho^2} + K$$

Chapitre 4 : Ex 4 :



$\vec{F} = -k\,\vec{OM}$

$\vec{OM} = r\,\vec{e_r}$

$\sum \vec{F}_{ext} + \vec{f}_{ie} + \vec{f}_{ic} = m\vec{\Gamma}(M/R)$

$\left.\begin{array}{l} \vec{f}_{ie} = -m\,\vec{\Gamma}_e(M) \\ \vec{f}_{ic} = -m\,\vec{\Gamma}_c(M) \end{array}\right\} \quad \begin{array}{l} \vec{P} = -mg\vec{K} \\ \vec{F} = -k(r\vec{I}) \end{array}$

$\vec{R} = R_1\vec{I} + R_2\vec{J} + R_3\vec{K}$

$P(\vec{R})_{R_1} = 0 = \vec{R}\cdot\vec{V}(M/R_1)$

$= \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \cdot \begin{pmatrix} \dot r \\ 0 \\ 0 \end{pmatrix} = \dot r R_1 = 0$

$\vec{V}(M/R_1) = \frac{d}{dt}(r\vec{I})_{R_1} = \dot r\,\vec{J}$

$\dot r R_1 = 0 \Rightarrow \begin{cases} \dot r = 0 & \dot r \neq 0 \\ R_1 = 0 & \vec{R} = R_2\vec{J} + R_3\vec{K} \end{cases}$

RFD

$\vec{\Gamma}_r(M) = \ddot r\,\vec{J}$

$\vec{\Gamma}_e(M) = 0 + \vec{0} + \vec{\Omega}(R_1/R_0) \wedge [\vec{\Omega}(R_1/R_0) \wedge \vec{OM}]$

$\vec{\Gamma}_a(O) \quad \overset{0}{\frac{d}{dt}}[\vec{\Omega}(R_1/R_0)] \wedge \vec{OM}$

$= w\vec{K} \wedge [w\vec{K} \wedge r\vec{I}] = -rw^2\vec{I}$

$\vec{\Gamma}_c(M) = 2\,\vec{\Omega}(R_1/R_0) \wedge \vec{V}_1 = 2w\vec{K} \wedge \dot r\vec{I}$

$= 2w\dot r\,\vec{J}$

$m\,\vec{\Gamma}_r(M) = \sum \vec{F}_{ext} + \vec{f}_{ie} + \vec{f}_{ic} + \vec{R} + \vec{F}$

$m\ddot r\vec{I} = -mg\vec{K} + R_2\vec{J} + R_3\vec{K} + mrw^2\vec{I}$

$- 2mw\dot r\,\vec{J} - Kr\vec{I}$

$\begin{cases} m\ddot r = mrw^2 - Kr & ① \\ 0 = R_2 - 2mw\dot r & ② \\ 0 = -mg + R_3 & ③ \end{cases}$

$\boxed{R_3 = mg}$

(1) $\Rightarrow \ddot r + r\left(\frac{k}{m} - w^2\right) = 0$

1er cas :

si $\frac{k}{m} - w^2 > 0$

Posons $W^2 = \frac{k}{m} - w^2$

$\ddot r + W^2 r = 0$

$\begin{cases} A\cos Wt + B\sin Wt \\ W = \sqrt{\frac{k}{m} - w^2} \end{cases}$

$r(0) = A = r_0$

$\dot r(0) = BW \Rightarrow B = 0$

$r(t) = r_0\cos Wt$

2eme cas :

si $\frac{k}{m} - w^2 < 0$

Posons $W^2 = w^2 - \frac{k}{m}$

$\ddot r - W^2 r = 0$

$X^2 - W^2 = 0 \Rightarrow X = \pm W$

$r(t) = Ae^{Wt} + Be^{-Wt}$

$r(0) = A + B = r_0$
$\dot r(0) = AW - BW = 0 \Leftrightarrow \begin{cases} A + B = r_0 \\ A - B = 0 \end{cases}$

$2A = r_0 \Rightarrow A = B = \frac{r_0}{2}$

$r(t) = \frac{r_0}{2}\left[e^{Wt} + e^{-Wt}\right]$

$r = r_0\,\mathrm{ch}\,Wt$

3eme cas :

$\frac{k}{m} - w^2 = 0$

$\ddot r = 0 \Rightarrow \dot r = cte = \dot r_0 = 0$

$\dot r = 0 \Rightarrow r = cte = r_0$

2) Equilibre relative

$m\vec{\Gamma_r}(M) = \vec{0} = m\ddot{r}\vec{I}$

$\ddot{r} = 0 \Rightarrow \dot{r} = cte = \dot{r}_0 = 0$

$\dot{r} = 0 \Rightarrow r = cte = r_0$

3) $R_N = \sqrt{R_2^2 + R_3^2}$

$R_3 = mg$

$R_2 = 2\omega\dot{r}$

1er cas :

$r = r_0 \cos Wt$

$\dot{r} = -r_0 W \sin Wt$

$R_2 = -2m\omega W r_0 \sin Wt$

$R_N = \sqrt{(mg)^2 + 4((m\omega)Wr_0 \sin Wt)^2}$

2ème cas

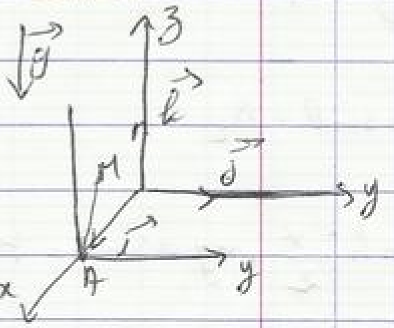
$r = r_0 \operatorname{ch} Wt$

$\dot{r} = r_0 W \operatorname{sh} Wt$

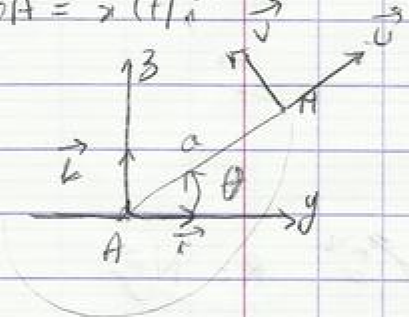
$R_N = \sqrt{(mg)^2 + 4(r_0 W \operatorname{sh} Wt)^2}$

Chapitre 5 :

Ex 1 :



$\vec{OA} = x(t)\vec{i}$



$\vec{F} = -mh^2\vec{OM}$

$\vec{P} = -mg\vec{k}$

$\vec{R} = -mN\vec{\mu}$

$\vec{U} = \frac{\vec{AM}}{\|\vec{AM}\|}$

$\vec{V} = \vec{i}\wedge\vec{U}$

$B(\vec{i},\vec{U},\vec{V})$

1) $\vec{V}(M/R_0) = \frac{d}{dt}\left[\vec{OM}\right]_{R_0}$

$= \frac{d}{dt}\left(\vec{OA} + \vec{AM}\right)_{R_0} = \frac{d}{dt}\left(x\vec{i} + a\vec{u}\right)_{R_0}$

$\dot{x}\vec{i} + a\left(\frac{d\vec{u}}{d\theta}\right)_{R_0}\left(\frac{d\theta}{dt}\right) = \dot{x}\vec{i} + a\dot{\theta}\vec{V}$

$\vec{\Gamma}(M/R_0) = \ddot{x}\vec{i} + a\ddot{\theta}\vec{V} - a\dot{\theta}^2\vec{u}$.

$\sum \vec{F}_{ext} = m\vec{\Gamma}(M/R_0)$

$\vec{P} + \vec{F} + \vec{R} = -mg\vec{k} - mh^2(x\vec{i} + a\vec{u})$

$- mN\vec{u}$

$\vec{k} = \cos\theta\vec{V} + \sin\theta\vec{u}$.

$$\begin{array}{l} \vec{i} \\ \vec{u} \\ \vec{V} \end{array} \begin{cases} m\ddot{x} = -mh^2 x \\ -ma\dot{\theta}^2 = -mh^2a - mN \\ ma\ddot{\theta} = mg\cos\theta \end{cases}$$

2)

$\vec{L}_0(M/R) = \vec{OM}\wedge m\vec{V}(M/R)$.

$$\begin{pmatrix} x \\ a \\ 0 \end{pmatrix}_B \wedge \begin{pmatrix} \dot{x} \\ 0 \\ a\dot{\theta} \end{pmatrix} = \begin{pmatrix} ma^2\dot{\theta} \\ -max\dot{\theta} \\ -ma\dot{x} \end{pmatrix}_B$$

$\frac{d}{dt}\left[\vec{L}_0(M/R_0)\right]_{R_0} = \sum \vec{M}_0(\vec{F}_{ext})$

$\frac{d}{dt}\left[\vec{L}_0(M/R_0)\right]_{R_0} = m\left[a^2\ddot{\theta}\vec{i} + (\cancel{-a\dot{x}\dot{\theta}} - ax\ddot{\theta})\vec{u} - ax\dot{\theta}^2\vec{V} - a\ddot{x}\vec{V} + \cancel{a\dot{x}\dot{\theta}\vec{u}}\right]$

$= m\left[a^2\ddot{\theta}\vec{i} - ax\ddot{\theta}\vec{u} + (-ax\dot{\theta}^2 - a\ddot{x})\vec{V}\right]$

$\sum \vec{M}_0(\vec{F}_{ext}) = \vec{M}_0(\vec{P}) + \vec{M}_0(\vec{F}) + \vec{M}_0(\vec{R})$

$\vec{M}_0(\vec{P}) = \vec{OM}\wedge\vec{P}$

$$\begin{pmatrix} x \\ a \\ 0 \end{pmatrix}_B \wedge \begin{pmatrix} 0 \\ -mg\sin\theta \\ -mg\cos\theta \end{pmatrix}_B = \begin{pmatrix} -mag\cos\theta \\ mgx\cos\theta \\ -mgx\sin\theta \end{pmatrix}_B.$$

$\vec{M}_0(\vec{F}) = \vec{0}$ car $\vec{F} // \vec{OM}$

$= \vec{OM}\wedge(-mh^2\vec{OM}) = \vec{0}$

$$\vec{M}_0(\vec{R}) = \begin{pmatrix} x \\ a \\ 0 \end{pmatrix}_B \wedge \begin{pmatrix} 0 \\ -mN \\ 0 \end{pmatrix}_B = \begin{pmatrix} 0 \\ 0 \\ -mNx \end{pmatrix}_B$$

$\sum \vec{M}_0 \vec{F}_{ext} = \frac{d}{dt}\left[\vec{L}_0(M/R_0)\right]_{R_0}$

$= ma^2\ddot{\theta} = -mga\cos\theta$

$\boxed{= ma\ddot{\theta} = -mg\cos\theta}$ (3) cqfd

3) $\sum \vec{F}_{ext} = -\vec{grad}\, E_p \Leftrightarrow$

$$P(\vec{F}_{ext}) = \frac{-d}{dt}[E_p]$$

$$P(\vec{F}_{ext}) = \sum \vec{F}_{ext} \cdot \vec{V}(M/R_0)$$

$$\begin{array}{l} \vec{i} \\ \vec{u} \\ \vec{v} \end{array}\begin{pmatrix} -mh^2x \\ -mg\sin\theta - mN - mh^2a \\ -mg\cos\theta \end{pmatrix}_B \cdot \begin{pmatrix} \dot{x} \\ 0 \\ a\dot{\theta} \end{pmatrix}_B$$

$$= mh^2 x\dot{x} - mga\dot{\theta}\cos\theta$$

$$= \frac{-d}{dt}\left[mh^2\frac{x^2}{2} + mga\sin\theta + cte\right].$$

$$E_p = \frac{mh^2}{2}x^2 + mga\sin\theta + C.$$

$$E_c(M/R_0) = \frac{1}{2}m(\dot{x}^2 + a^2\dot{\theta}^2)$$

$$E_c + E_p = E_m = cte = \frac{1}{2}m(\dot{x}^2 + a^2\dot{\theta}^2) + \frac{mh^2}{2}x^2$$

$$+ mga\sin\theta = C.$$

à $t = 0 \Rightarrow cte = 0 \Rightarrow C = 0.$

4) $W_{A(x=0,\ \theta=0)}^{B(x=0,\ \theta=\pi/2)}$

$$= E_p(A) - E_p(B)$$

$$E_p(x=0, \theta=0) = C$$

$$E_p(x=0, \theta=\frac{\pi}{2}) = mga + C.$$

$$W_A^B = mga < 0$$

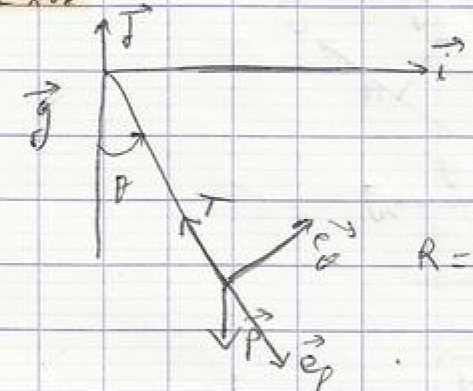
$$\begin{cases} m\ddot{x} = -mh^2x\ (1) \times \dot{x} \\ -ma\dot{\theta}^2 = -mh^2a - mN - mg\sin\theta \\ ma\ddot{\theta} = -mg\cos\theta\ (3) \cdot \dot{\theta}a \end{cases}$$

$$m\frac{\dot{x}^2}{2} = -mh^2\frac{x^2}{2} + K_1$$

$$ma\frac{\dot{\theta}^2}{2} = -mga\sin\theta + K_2$$

$$m\left(\frac{\dot{x}^2}{2} + a^2\dot{\theta}^2\right) + mh^2\frac{x^2}{2} + mga\sin\theta = K.$$

Ex 2 :



$R = N\vec{e}_\rho$

L.F.D : $\vec{P} + \vec{T} + \vec{R} = m\vec{\Gamma}(M)_R$

$$\vec{OM} = l\vec{e}_\rho$$

$$\vec{V}(M/R) = l\dot{\theta}\vec{e}_\theta$$

$$\vec{\Gamma}(M/R) = l\ddot{\theta}\vec{e}_\theta - l\dot{\theta}^2\vec{e}_\rho$$

$$\begin{cases} mg\cos\theta - T + N = -ml\dot{\theta}^2 \\ -mg\sin\theta = ml\ddot{\theta} \\ 0 = 0 \end{cases}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\frac{d}{dt}\left[\vec{L}_O(M/R)\right]_R = \sum \vec{M}_O(\vec{F}_{ext})$$

$$\vec{M}_O(\vec{P}) = \vec{OM} \wedge \vec{P} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} mg\cos\theta \\ -mg\sin\theta \\ 0 \end{pmatrix}$$

$$= -mg\,l\sin\theta\,\vec{K}$$

$$\vec{M}_O(\vec{T}) = \vec{0} \qquad \vec{T} \parallel \vec{OM}$$

$$\vec{M}_O(\vec{R}) = \vec{0} \qquad \vec{R} \parallel \vec{OM}$$

$$\vec{L}_O(M/R) = \vec{OM} \wedge m\vec{V}(M/R)$$

$$= l\vec{e}_r \wedge ml\dot{\theta}\vec{e}_\theta$$

$$= ml^2\dot{\theta}\vec{K}$$

$$\frac{d}{dt}\left[\vec{L}_0(M)_R\right]_R = ml^2\ddot{\theta}\vec{K}$$

$$-\cancel{m}gl\sin\theta = \cancel{m}l^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

3) $P(\vec{F}_{ext}) = \frac{-d}{dt}[E_p]$

$$P(\Sigma\vec{F}_{ext}) = \sum\vec{F}_{ext}\cdot\vec{V}(M)_R$$

$$\begin{pmatrix} mg\cos\theta + N - T \\ -mg\sin\theta \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ l\dot{\theta} \\ 0 \end{pmatrix}$$

$$P(\vec{F}_{ext}) = -mgl\dot{\theta}\sin\theta$$

$$= \frac{-d}{dt}\left[-mgl\cos\theta + cte\right]$$

$$E_m = E_c + E_p = cte$$

$$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = cte$$

$$\cancel{m}l^2\cancel{\dot{\theta}}\ddot{\theta} + \cancel{m}gl\cancel{\dot{\theta}}\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$\theta \leq 10°$ $\quad \ddot{\theta} + \frac{g}{l}\theta = 0$

$$\Rightarrow \ddot{\theta} + \omega^2\theta = 0$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$